NOTE: Problems not specifically asking for FBD, EOM, etc, in the question, yet providing points to them as part of the solution, are to have points redistributed within the question should students provide a solution that does not include them.

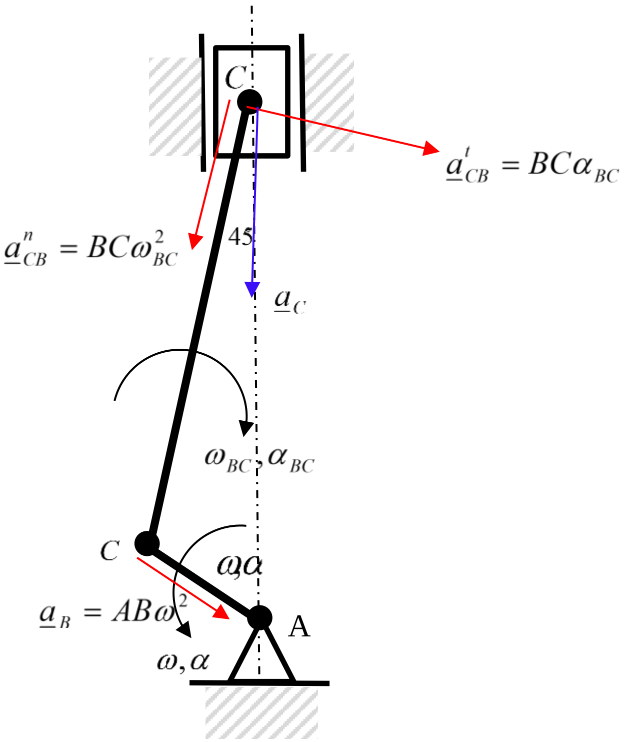
Q1

Figure Q1 shows a slider-crank mechanism in which a motor drives crank AB about a fixed axis at A causing connecting rod BC to exhibit general plane motion and the slider at C to translate in the vertical direction. The relevant dimensions of the links are shown in the figure. At the instant shown angle $\gamma = 13.63^{\circ}$, the crank AB has constant angular speed $\omega = 100\,\text{rpm}$, the connecting rod BC has angular velocity $\omega_{BC} = 2.5405\,\text{rad/s}$ and the slider has velocity $v = 0.9209\,\text{m/s}$.

(i) The acceleration of C is calculated using the following vector equation:

$$\underline{a}_C = \underline{a}_B + \underline{a}_{CB}^n + \underline{a}_{CB}^t \quad (2) \quad [1]$$

Where



The angular velocity of the crank AB in rad/s is: $100x2\pi/60 = 10.4720\text{rad/s}$

[1]

The magnitudes are:

$a_B = AB\omega^2 = (0.1)(10.4720)^2 = 10.9663\text{m/s}^2$ [1]

$a_{CB}^n = BC\omega_{BC}^2 = (0.3)(2.5405)^2 = 1.9362\text{m/s}^2$ [1]

$a_{CB}^t = BC\alpha_{BC} = (0.3)\alpha_{BC}$ [1]

Directions of acceleration components (see diagram)

[3]

Resolving equation (2) parallel to BC gives:

$$a_C \cos\gamma = AB\omega^2 \cos(45+\gamma) + a_{CB}^n$$ [3]

Re-arranging and substituting values gives:

$$a_C = \frac{AB\omega^2 \cos(45+\gamma) + a_{CB}^n}{\cos\gamma} = \frac{10.9663\cos(45+13.633)+1.9362}{\cos(13.633)} = 7.8660\text{m/s}^2$$

[1]

Slider accelerates vertically downwards [1]

Resolving equation (2) vertically gives:

$$a_C = AB\omega^2 \cos(45) + a_{CB}^n \cos\gamma + a_{CB}^t \sin\gamma$$ [2]

Rearranging and substituting values gives:

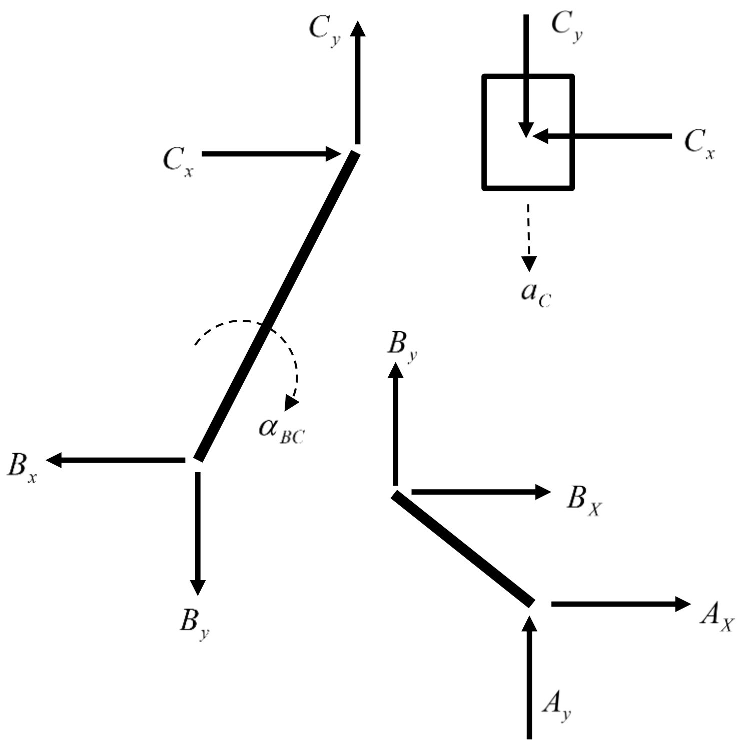
$$\alpha_{BC} = \frac{a_C - AB\omega^2 \cos(45) - a_{CB}^n \cos\gamma}{BC\sin\gamma} = \frac{7.8579-10.9663\cos(45)-1.9362\cos(13.633)}{(0.3)\sin(13.633)}$$

$$= -25.15\text{rad/s}^2$$

[2]

i.e. the angular acceleration of link BC is 25.5rad/s^2 in the anti-clockwise direction
[1]
[Total part a) = 18 marks]

(ii) Free body diagrams



[Total part b) = 6 marks]

(iii) Using D'Alembert's Principle for the piston gives:

$$C_y - m_{piston} a_c = 0$$ [1]

Re-arranging and substituting values gives:

$$C_y = 1\text{x}7.8860 = 7.8660\ \text{N}$$ [1]

Using D'Alembert's Principle for the connecting rod, taking moments about B gives:

$$-J_B \alpha_{BC} - C_x BC \cos\gamma + C_y BC \sin\gamma = 0$$ [2]

Re-arranging and substituting values gives:

$$C_x = \frac{C_y BC \sin\gamma - J_B \alpha_{BC}}{BC \cos\gamma} = \frac{7.8860\text{x}0.3\text{x}\sin(13.633) + 2x25.15}{0.3\text{x}\cos(13.633)} = 174.44\text{N}$$

[2]

[Total part b) = 6 marks]

Q2

1. The block diagram for an open loop system is shown in figure 1.

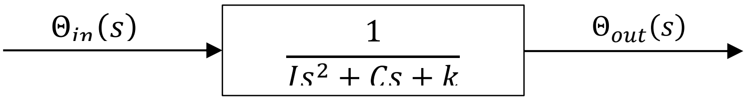


Figure 1

(i) For J=2 and k=6, calculate the value of C that will result in a critically damped system.

[5 marks]

Solution:

To be critically damped ($\gamma = 1$), $Js^2 + Cs + k = J(s^2 + 2\gamma\omega s + \omega^2)$ must be such that if $\frac{k}{j} = \omega^2$ then $\frac{C}{J} = 2\gamma\omega = 2\omega$ and

$$C = 2\sqrt{kJ}$$

Hence $C = 2\sqrt{12}$ =6.93

(ii) Calculate the natural frequency of the system.

[5 marks]

Solution:

From part (a), $Js^2 + Cs + k = J(s^2 + 2\gamma\omega s + \omega^2)$ so $\frac{k}{j} = \omega^2$ and

$$\omega = \sqrt{3} = 1.73$$

(iii)For a step input given by $\Theta_{in}(s) = {}^{2}/_{s}$, calculate the magnitude of the steady state error defined as:

$$E_{ss} = \lim_{t\to\infty}\big(\theta_{in}(t) - \theta_{out}(t)\big)$$

[5 marks]

Hint:

Function	Laplace Transform
$1 - \omega t e^{-\omega t}$	$\frac{\omega^2}{s(s^2 + 2\omega s + \omega^2)}$

Solution:

Magnitude can be calculated either from reverse Laplace transforms or using the final value theorem.

Reverse Laplace transforms:

$$\Theta_{out}(s) = \Theta_{in}(s)\frac{1}{Js^2 + Cs + k} = \frac{2}{s(2s^2 + 6.93s + 6)} = \frac{1}{s(s^2 + 3.47s + 3)}$$

$$\frac{1}{s(s^2 + 3.47s + 3)} = \frac{1}{3}\left(\frac{\omega^2}{s(s^2 + 2\omega s + \omega^2)}\right)$$

Where $\omega = \sqrt{3}$ and $\gamma = 1$

Becomes

$$\theta_{out}(t) = \frac{1}{3}(\,1 - \omega t e^{-\omega t})$$

Hence, steady state error is given by:

$$\lim_{t\to\infty}\big(\theta_{in}(t) - \theta_{out}(t)\big) = 2 - \frac{1}{3} = \frac{5}{3} or 1.67$$

By final value theorem:

$$\lim_{t\to\infty}\left(\theta_{out}(t)\right) = \lim_{s\to 0}\left(s\Theta_{out}(s)\right) = \lim_{s\to 0}\left(\frac{2s}{s(2s^2+6.93s+6)}\right) = \frac{1}{3}$$

Hence steady state error is

$$\lim_{s\to 0}\left(s\Theta_{in}(s) - s\Theta_{out}(s)\right) = \lim_{s\to 0}\left(\frac{2s}{s} - \frac{2s}{s(2s^2+6.93s+6)}\right) = 2 - \frac{1}{3} = 1.67$$

(5 marks for correct answer, 3 marks for a correct calculation of $\lim_{t\to\infty}\left(\theta_{out}(t)\right)$ if the steady state error is incorrect.

The open loop system in Figure Q2 is then incorporated into a closed loop system as shown in Figure Q2.

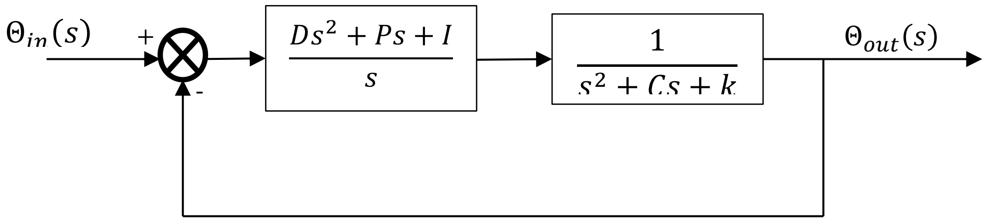


Figure Q3

(iv) What is the overall transfer function, $G(s) = \frac{\Theta_{out}(s)}{\Theta_{in}(s)}$ for the system shown in Figure Q3?

[5 marks]

Solution:

$$\Theta_{out}(s) = \left(\Theta_{in}(s) - \Theta_{out}(s)\right)\frac{Ds^2+Ps+I}{s(Js^2+Cs+k)}$$

$$\Theta_{out}(s)\left(1 + \frac{Ds^2+Ps+I}{s(Js^2+Cs+k)}\right) = \Theta_{in}(s)\frac{Ds^2+Ps+I}{s(Js^2+Cs+k)}$$

$$\Theta_{out}(s)(s(Js^2+Cs+k) + Ds^2+Ps+I) = \Theta_{in}(s)(Ds^2+Ps+I)$$

$$\frac{\Theta_{out}(s)}{\Theta_{in}(s)} = \frac{Ds^2+Ps+I}{Js^3+(C+D)s^2+(P+k)s+I}$$

(v) For the values of P, D, I, J and k given in table 1, calculate the range of values of C for which the system will be stable.

P	D	I	J	k
2	1	10	2	6

Table 1

[10 marks]

Routh-Hurwitz Criteria: No change of sign so C>-1; (worth 4)

Q3

FIGURE Q# shows a rigid bar AB which pivots about fixed point O.

DATA
I_O = 0.5 kg m^2
L_1 = 0.4 m
L_2 = 0.6 m
K_1 = 1000 N/m
K_2 = 1000 N/m
c = 50 Ns/m

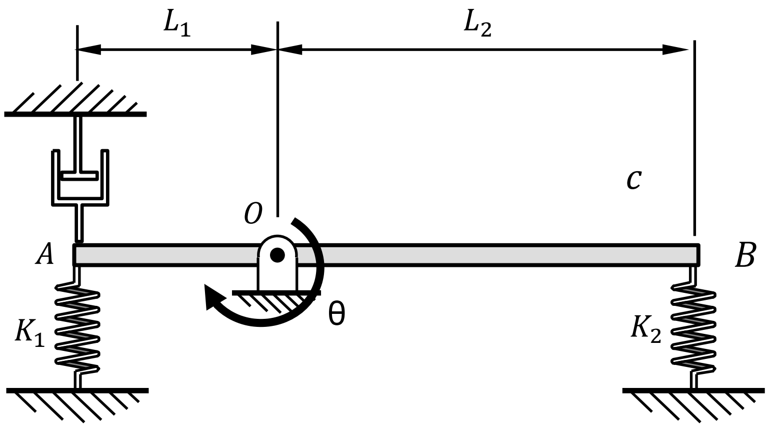
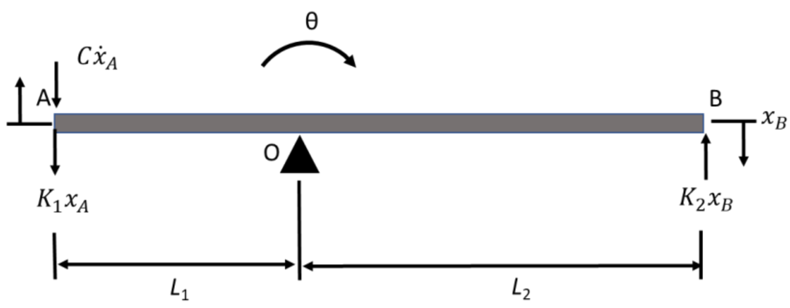
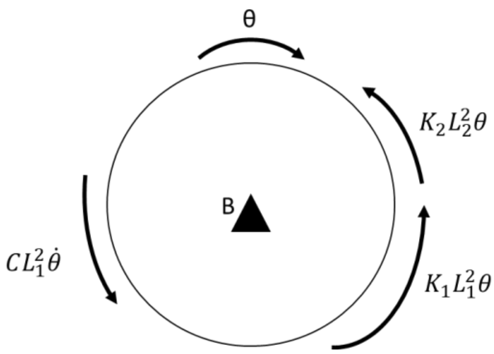


Figure Q#

(i) Draw a fully annotated Free Body Diagram for the bar. [4]

Don't forget the spring and damper are linear, so in the FBD below they show as linear forces, but when transferring to EOM in part b) they'll need to become torques.





(ii) Derive the equation of motion governing rotation of the beam θ(t) about pivot O. [3]

Relationships for linear displacements at springs to rotational displacements (students may have included this in part A).

$$x_A = L_1\theta$$
$$\dot{x}_A = L_1\dot{\theta}$$

$$x_B = L_2\theta$$

Keeping in mind the forces shown in the diagram for the spring and damper are linear, so they will have to be converted to torques the EOM now becomes…

$$I_o\ddot{\theta} = -CL_1^2\dot{\theta} - K_2L_2^2\theta - K_1L_1^2\theta$$

$$I_o\ddot{\theta} + CL_1^2\dot{\theta} + (K_1L_1^2 + K_2L_2^2)\theta = 0$$

$$0.5\ddot{\theta} + 8\dot{\theta} + 520\theta = 0$$

[1pt for newton's; 1 pt for in correct form; 1 pt for correct derivation]

(iii) Calculate the undamped natural frequency ωn and damping ratio ʓ for the system.

[4]

[2 pts each]

$$\omega_n = \sqrt{\frac{K_1L_1^2 + K_2L_2^2}{I_o}} = 32.25\frac{rad}{s} = 5.133Hz$$

$$\gamma = \frac{CL_1^2}{2\sqrt{(K_1L_1^2+K_2L_2^2)I_o}} = 0.25$$

(iv) The end B is lifted up by 0.05 m and the bar is then released from rest. Determine the resulting transient angular displacement at O as a function of time, $\theta_{tr}(t)$.

[6]

The damping ratio is less than 1, so this is an underdamped system and the following equations apply from the formula sheet.

$$z(t)_{tr} = e^{-\gamma\omega_n t}[B_1 \cos(\Omega_n t) + B_2 \sin(\Omega_n t)]$$

$$\dot{z}(t)_{tr} = B_1 e^{-\gamma\omega_n t}[-\Omega_n \sin(\Omega_n t) - \gamma\omega_n \cos(\Omega_n t)] + B_2 e^{-\gamma\omega_n t}[\Omega_n \cos(\Omega_n t) - \gamma\omega_n \sin(\Omega_n t)]$$

Where

$$\Omega_n = \omega_d = \omega_n\sqrt{1-\gamma^2} = 31.24\frac{rad}{s}$$

[1 pt]

For initial conditions where x_o=0.05m at point A the rotation will be $\theta_o = \frac{0.05}{L_2} = 0.0833\, rad$ and $t = 0$.

$$\theta(t)_{tr} = 0.0833 = 1[B_1 * 1 + B_2 * 0)]$$
$$B_1 = 0.08333$$

[1]

For initial conditions where $\dot{\theta}(0)_{tr} = 0$ and $t = 0$.

$$\dot{\theta}(t)_{tr} = 0 = 0.1[-\gamma\omega_n] + B_2[\Omega_n]$$
$$B_2 = \frac{0.1\gamma\omega_n}{\omega_n\sqrt{1-\gamma^2}} = 0.0213$$

[3]

Therefore

$$\theta(t)_{tr} = e^{-8t}[0.08333 \cos(31.24t) + 0.02134 \sin(31.24t)]$$

[1]

(v) What frequency will this system vibrate at after being released from rest? [2]

Should have been calculated above, but for completeness to reporting in Hz the vibration frequency of the system will be:

$$\Omega_n = \omega_d = \omega_n\sqrt{1-\gamma^2} = 31.24\frac{rad}{s} = 4.97\ Hz$$

[only 1pt if not in Hz]

(vi) For the subsequent vibration, calculate the speed of end B when it first passes through the equilibrium position. [6; either solution is ok]

Using the exact solution...

$$\theta(t)_{tr} = 0$$

When

$$0.08333\,cos(\,31.24t) + \;0.02134\,sin(\,31.24t) = 0$$

Or

$$tan(31.24t) = \frac{0.08333}{0.02134}$$

Therefore

t=0.0423 s

Hence

$$\dot{\theta} = -1.917\,rad/s$$
$$\dot{x}_B = L_2\dot{\theta} = -1.150\,m/s$$

Using the approximate solution...

The approximate solution is that this occurs at ¼ of the cycle. E.g. when

$$\Omega_n t = \frac{\pi}{2}$$

Using this approximation and the equations from part (iv) for Ω_n and $\theta(t)_{tr}$ results in...

$$t = 0.0503s$$

$$\dot{\theta} = -1.856\,rad/s$$

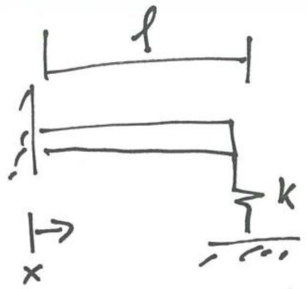
$$\dot{x}_B = L_2\dot{\theta} = -1.113\,m/s$$

Q4

Figure Q# shows a uniform beam of length, L, density ρ, Young's modulus of elasticity E, and second moment of area I. One end of the beam is clamped, while the other end is connected to a spring of stiffness, k, as shown.

(i) Using boundary conditions determine the generalized matrix, [Z]{C}={0}, that could be used to solve for undamped natural frequencies and mode shapes of the beam. Note: It is not required for you to solve for the constants {C}, only show the generalized matrix and terms contained in [Z]. [13]

Solutions by hand due to time constraints.



At the wall ($x=0$)

$y(0)=0$

$0 = C_1 \sin 0 + C_2 \cos 0 + C_3 \sinh 0 + C_4 \cosh 0$

$C_2 + C_4 = 0$ eqn 1

+2

slope $= \frac{dy}{dx} = 0$

$0 = \lambda C_1 \cos 0 - \lambda C_2 \sin 0 + \lambda C_3 \cosh 0 + \lambda C_4 \sinh 0$

$0 = \lambda C_1 + \lambda C_3$

eqn 2

+2

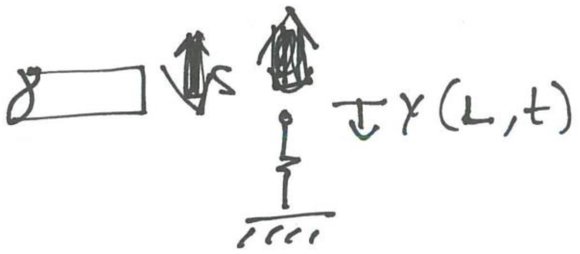
At the spring ($x=L$)

$M = \frac{d^2y}{dx^2} = 0$

$0 = -\lambda^2 C_1 \sin \lambda L - \lambda^2 C_2 \cos \lambda L + \lambda^2 C_3 \sinh \lambda L + \lambda^2 C_4 \cosh \lambda L$

eqn 3

+2



$S = -ky \qquad S = EI \left.\frac{\partial^3 y}{\partial x^3}\right|_{x=L}$

$-ky = EI \frac{\partial^3 y}{\partial x^3}$

$EI \frac{\partial^3 y}{\partial x^3} + ky = 0$ eqn 4

+4

$EI\left(-\lambda^3 C_1 \cos(\lambda L) + \lambda^3 C_2 \sin(\lambda L) + \lambda^3 C_3 \cosh(\lambda L) + \lambda^3 C_4 \sinh(\lambda L)\right)$

$+$

$k\left(C_1 \sin(\lambda L) + C_2 \cos(\lambda L) + C_3 \sinh(\lambda L) + C_4 \cosh(\lambda L)\right) = 0$

$\Downarrow$

$(k \sin \lambda L - EI\lambda^3 \cos \lambda L)\, C_1$

$+$

$(k \cos \lambda L + EI\lambda^3 \sin \lambda L)\, C_2$ eqn 4

+2

$+$

$(k \sinh \lambda L + EI\lambda^3 \cosh \lambda L)\, C_3$

$+$

$(k \cosh \lambda L + EI\lambda^3 \sinh \lambda L)\, C_4$

$= 0$

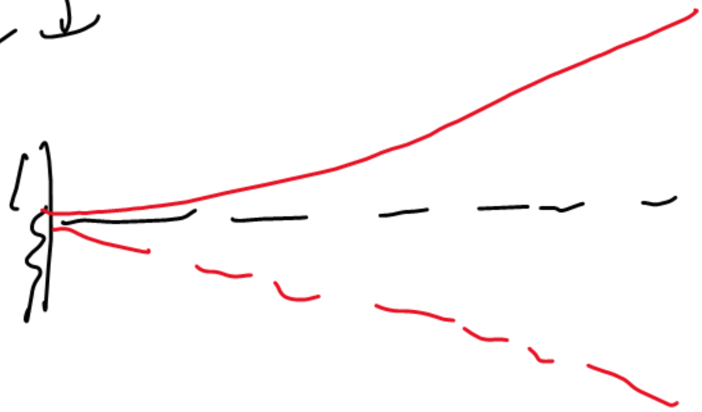
$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
\lambda & 0 & \lambda & 0 \\
-\lambda^2 \sin\lambda L & -\lambda^2 \cos\lambda L & \lambda^2 \sinh\lambda L & \lambda^2 \cosh\lambda L \\
k\sin\lambda L - EI\lambda^3\cos\lambda L & k\cos\lambda L + EI\lambda^3\sin\lambda L & k\sinh\lambda L + EI\lambda^3\cosh\lambda L & k\cosh\lambda L + EI\lambda^3\sinh\lambda L
\end{bmatrix}
\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

+1

(ii) Provide simple sketches for the first two modes of vibration. [2]

I couldn't get my drawing pad to work, so excuse the poor drawing skills, but you should get the idea.

mode 1



mode 2